MATH2068: Mathematical Analysis II
Home Test 1 **5:00pm, 24 Mar 2023**

Important Notice:

♣ The answer paper **Must be submitted before 25 Mar 2023 at 5:00 pm.**
♠ The answer paper MUST BE sent to the CU Blackboard.
✠ The answer paper **Must include your name and student ID in each page.**

Answer ALL Questions

1. **(30 points)**

   (a) Let $\alpha > -1$ and let f be a monotone function on $(0, 1]$. Prove or disprove the following statements: if the improper integral $\int_0^1 x^\alpha f(x)dx$ exists, then $\lim_{x\to 0} x^{\alpha+1} f(x) = 0$.

   (b) Show that if f is a Riemann integrable function over $[0, 1]$, then the set of all continuous points of f is dense in $[0, 1]$.
   Is the above assertion still true if f is defined on $\mathbb{R}$ so that the improper integral $\int_{-\infty}^{\infty} f(x)dx$ is convergent? That is, whether the set of all continuous points of f is dense in $\mathbb{R}$ in this case.

   (c) If $f \in R[0, 1]$, does the following statement hold: $\int_0^1 (f(x))^2 dx = 0$ if and only if $f(c) = 0$ for all continuous point c of f.

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2. **(30 points)** Let $u : \mathbb{R}^n \to \mathbb{R}$ be a sub-additive positive homogenous function, that is, it satisfies the conditions: $u(\mathbf{x}+\mathbf{y}) \le u(\mathbf{x}) + u(\mathbf{y})$ and $u(t\mathbf{x}) = tu(\mathbf{x})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $t \ge 0$.

   (a) Show that for every $\mathbf{x}, \mathbf{h} \in \mathbb{R}^n$, the function $t \in \mathbb{R} \mapsto u(\mathbf{x} + t\mathbf{h})$ is convex. From this, show that the function $t \mapsto \frac{u(\mathbf{x}+t\mathbf{h})-u(\mathbf{x})}{t}$ is monotone on $\mathbb{R} \setminus \{0\}$.

   (b) By using Part (a), show that for every $\mathbf{x}, \mathbf{h} \in \mathbb{R}^n$,
   $\lim\limits_{t\to 0} \dfrac{u(\mathbf{x}+t\mathbf{h}) - u(\mathbf{x})}{t}$ exists if and only if $\lim\limits_{t\to 0} \dfrac{u(\mathbf{x}+t\mathbf{h}) + u(\mathbf{x}-t\mathbf{h}) - 2u(\mathbf{x})}{t} = 0.$

   (c) Fix $\mathbf{x} \in \mathbb{R}^n$ and assume that $u'_{\mathbf{x}}(\mathbf{h}) := \lim\limits_{t\to 0} \dfrac{u(\mathbf{x}+t\mathbf{h}) - u(\mathbf{x})}{t}$ exists for all $\mathbf{h} \in \mathbb{R}^n$. Show that the function $u'_{\mathbf{x}} : \mathbb{R}^n \to \mathbb{R}$ is linear.

$* * *$ **END OF PAPER** $* * *$